NOTE ON THE X_1-LAGUERRE ORTHOGONAL POLYNOMIALS.

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Abstract. This note supplements the results in the paper on X_1-**Laguerre** orthogonal polynomials written by David Gómez-Ullate, Niky Kamran and Robert Milson.

1. Introduction

This note reports on, the X_1-**Laguerre polynomials**, one of the two new sets of orthogonal polynomials considered in the papers [3] and [4], written by David Gómez-Ullate, Niky Kamran and Robert Milson. The other set is named the X_1-**Jacobi polynomials** and is discussed, in similar terms, in the note [2].

These two papers are remarkable and invite comments on the results therein which have yielded new examples of Sturm-Liouville differential equations and their associated differential operators.

The two sets of these orthogonal polynomials are distinguished by:

(*i*) Each set of polynomials is of the form $\{P_n(x) : x \in \mathbb{R} \text{ and } n \in \mathbb{N} \equiv \{1, 2.3 \ldots\}\}$ with $\deg(P_n) = n$; that is there is no polynomial of degree 0.

(*ii*) Each set is orthogonal and complete in a weighted Hilbert function space.

(*iii*) Each set is generated as a set of eigenvectors from a self-adjoint Sturm-Liouville differential operator.

2. X_1-Laguerre polynomials

These polynomials and the associated differential equation are detailed in [3, Section 2].

In [3, Section 2, (21)] the second-order linear differential equation concerned is given as

$$-xy''(x) + \left(\frac{x-k}{x+k}\right)((x+k+1)y'(x) - y(x)) = \lambda y(x) \text{ for all } x \in (0,\infty) \tag{2.1}$$

where the parameter $\lambda \in \mathbb{C}$ plays the role of a spectral parameter for the differential operators defined below, and the parameter $k \in (0,\infty)$.

This equation (2.1) is not a Sturm-Liouville differential equation; such equations take the form, in this case taking the interval to be (o,∞),

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \text{ for all } x \in (0,\infty), \tag{2.2}$$

but can be transformed into this form on using the information in [3, Section 2]. In particular let the coefficients p_k, q_k, w_k be defined as follows;

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(i) $p_k, q_k, w_k : (0,\infty) \to \mathbb{R}$

(ii)

$$p_k(x) := \frac{x^k}{(x+k)^2}\exp(-x) \text{ for all } x \in (0,\infty) \tag{2.3}$$

(iii)

$$q_k(x) := -\frac{(x-k)x^k}{(x+k)^3}\exp(-x) \text{ for all } x \in (0,\infty) \tag{2.4}$$

$$w_k(x) := \frac{x^k}{(x+k)^2}\exp(-x) \text{ for all } x \in (0,\infty). \tag{2.5}$$

Let the Sturm-Liouville differential expression M_k have the domain

$$D(M_k) := \{f : (0,\infty) \to \mathbb{C} : f^{(r)} \in AC_{\text{loc}}(0,\infty) \text{ for } r = 0,1\} \tag{2.6}$$

and be defined by, for all $f \in D(M_k)$,

$$M_k[f](x) := -(p_k(x)f'(x))' + q_k(x)f(x) \text{ for almost all } x \in (0,\infty). \tag{2.7}$$

Now define the Sturm-Liouville differential equation by, for all $k \in (0,\infty)$,

$$M_k[y](x) = \lambda w_k(x)y(x) \text{ for all } x \in (0,\infty) \tag{2.8}$$

where $\lambda \in \mathbb{C}$ is a complex valued spectral parameter.

For an account of Sturm-Liouville theory of differential operators and equations, see [1, Sections 2 to 6].

It is important to notice that the differential equation (2.8) is equivalent to, and is derived from the differential equation (2.1), see again [3, Section 2.2, (22a)].

The differential equation (2.8) is to be studied in the Hilbert function space $L^2((0,\infty);w_k)$.

The symplectic form for M_k is defined by, for all $k \in (0,\infty)$ and for all $f,g \in D(M_k)$,

$$[f,g]_k(x) := f(x)(p_k\overline{g}')(x) - (p_kf')(x)\overline{g}(x) \text{ for all } x \in (0,\infty). \tag{2.9}$$

The maximal operator $T_{k,1}$ is defined by, for all $k \in (0,\infty)$,

$$\left\{\begin{array}{ll} (i) & T_{k,1} : D(T_{1,k}) \subset L^2((0,\infty);w_k) \to L^2((0,\infty);w_k) \\ (ii) & D(T_{k,1}) := \{f \in D(M_k) : f, w^{-1}M_k[f] \in L^2((0,\infty);w_k)\} \\ (iii) & T_{k,1}f := w^{-1}M_k[f] \text{ for all } f \in D(T_{1,k}). \end{array}\right. \tag{2.10}$$

All self-adjoint differential operators in $L^2((0,\infty);w_k)$ generated by M_k are given by restrictions of the maximal operator $T_{k,1}$; these restrictions are determined by placing boundary conditions at the endpoints 0 and ∞, on the elements of $D(T_{k,1})$. The number and type of boundary conditions depends upon the endpoint classification of M_k in $L^2((0,\infty);w_k)$; see [1, Section 5].

For the endpoint classification of the differential expression M_k in $L^2((0,\infty);w_k)$ we have the results, see again [1, Section 5];

(i) At 0^+ the classification is:

(2.11)

For all $k \in (0,3]$	limit-circle non-oscillatory
For all $k \in (3,\infty)$	limit-point.

(ii) At $+\infty$ the classification is:

(2.12)

For all $k \in (0,\infty)$	limit point.

To establish these properties we have the following results:

(1) For $\lambda = 0$ the function

$$\varphi_1(x) := x + k + 1 \text{ for all } x \in [0,\infty), \tag{2.13}$$

is a solution of the differential equation (2.8), for all $k \in (0,\infty)$; see [3, Section 2, (14)].

(2) We have $\varphi_1 \in L^2((0,\infty); w_k)$ for all $k \in (0,\infty)$.

(3) For $\lambda = 0$ the function

$$\varphi_2(x) := \varphi_1(x) \int_1^x \frac{1}{\varphi_1^2(t) p_k(t)}\, dt \text{ for all } x \in (0,\infty), \tag{2.14}$$

is a solution of the differential equation (2.8), for all $k \in (0,\infty)$; φ_2 is independent of φ_1.

(4) Asymptotic analysis shows that

$$\boxed{\varphi_1 \in L^2((0,\infty); w_k) \text{ for all } k \in (0,\infty)} \tag{2.15}$$

and

(2.16)

$\varphi_2 \notin L^2([1,\infty); w_k)$ for all $k \in (0,\infty)$
$\varphi_2 \in L^2((0,1]; w_k)$ for all $k \in (0,3]$
$\varphi_2 \notin L^2((0,1]; w_k)$ for all $k \in (3,\infty)$.

The endpoint classifications (2.11) and (2.12) follow from the results items 1 to 4 above; see [1, Section 5.].

We can now define the restriction A_k of the maximal operator $T_{k,1}$, see (2.10), which is self-adjoint in the Hilbert function space $L^2((0,\infty); w_k)$, and which has the X_1-Laguerre polynomials as eigenvectors. To obtain this result it is essential:

(i) To apply the general theory of such restrictions as given in the Naimark text [5, Chapter V, Sections 17 and 18].

(ii) To apply the detailed results on the properties of the X_1-Laguerre polynomials given in [3, Section 2].

At any limit-point endpoint no boundary condition is required; at the limit-circle endpoint 0^+ the boundary condition for any $f \in D(T_{k,1})$ takes the form

$$\lim_{x \to 0+} \left[f, \alpha_1\varphi_1 + \alpha_2\varphi_2\right](x) = 0, \tag{2.17}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$. Since the X_1-Laguerre polynomials are to be in the domain of the operator A_k we take $\alpha_1 = 1$ and $\alpha_2 = 0$.

Thus the domain $D(A_k)$ of our self-adjoint operator A_k restriction of the maximal operator T_k is defined as follows:

(*i*) For $k \in (0,3]$

$$D(A_k) := \{f \in D(T_{k,1}) : \lim_{x\to 0+} [f,\varphi_1](x) = 0\} \tag{2.18}$$

and

$$A_k f := w_k^{-1} M_k[f] \text{ for all } f \in D(A_k). \tag{2.19}$$

(*ii*) For $k \in (3,\infty)$

$$D(A_k) := D(T_{k,1}) \tag{2.20}$$

and

$$A_k f := w_k^{-1} M_k[f] \text{ for all } f \in D(A_k). \tag{2.21}$$

The spectrum and eigenvectors of $A_{\alpha,\beta}$ can be obtained from the results given in [3, Section 2]. The spectrum of $A_{\alpha,\beta}$ contains the sequence $\{\lambda_n = n : n \in \mathbb{N}_0\}$; the eigenvectors are given by $\{\hat{L}_{n+1}^{(k)} : n \in \mathbb{N}_0\}$, the X_1-**Laguerre** orthogonal polynomials.

Remark 2.1. (*i*) The notation $\lambda_n = n$ for all $n \in \mathbb{N}_0$ makes good comparison with the eigenvalue notation for the classical Laguerre polynomials; this sequence is independent of the parameter $k \in (0,\infty)$.

(*ii*) We note that $\hat{L}_{n+1}^{(k)}$ is a polynomial of degree $n+1$ for all $n \in \mathbb{N}_0$ and all $k \in (0,\infty)$.

(*iii*) Note that for $k \in (0,3]$, when the limit-circle condition holds at 0^+, it is essential to check that the polynomials $\{\hat{L}_{n+1}^{(k)}\}$ all satisfy the boundary condition at 0^+ as required in (2.18), *i.e.*

$$\lim_{x\to 0+} \left[\hat{L}_{n+1}^{(k)}, \varphi_1\right](x) = 0 \text{ for all } n \in \mathbb{N}_0. \tag{2.22}$$

This result follows since, using (2.13),

$$\begin{aligned}\left[\hat{L}_{n+1}^{(k)}, \varphi_1\right](x) &= p_k(x)\left[\hat{L}_{n+1}^{(k)}(x)\varphi_1'(x) - \hat{L}_{n+1}^{(k)\prime}\varphi_1(x)\right] \\ &= \frac{x^k}{(x+k)^2}\exp(-x)\left[\hat{L}_{n+1}^{(k)} - \hat{L}_{n+1}^{(k)\prime}(x+k+1)\right] \\ &= \mathcal{O}(x^k) \text{ as } x \to 0^+.\end{aligned}$$

It is shown in [3, Section 3, Proposition 3.3] that the sequence of polynomials

$$\left\{\hat{L}_{n+1}^{(k)} : n \in \mathbb{N}_0\right\}$$

is orthogonal and dense in the space $L^2((0,\infty); w_k)$, for all $k \in (0,\infty)$. This result implies that for all $k \in (0,\infty)$ the spectrum of the operator A_k consists entirely of the sequence of eigenvalues $\{\lambda_n : n \in \mathbb{N}_0\}$; from the spectral theorem for self-adjoint operators in Hilbert space it follows that no other point on the real line $\mathbb{R}$ can belong to the spectrum of A_k.

Remark 2.2. It is to be noted that whilst the Hilbert space theory as given in [1] and [5] provides a precise definition of the self-adjoint operator A_k, the information about the particular spectral properties of A_k are to be deduced from the classical analysis results in [3]. Without these results it would be very difficult to deduce the spectral properties of the self-adjoint operator A_k, as defined above, in the Hilbert function space $L^2((0,\infty);w_k)$.

References

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